# Unitary Representations via The Orbit Method

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Isaac Newton encrypted his discoveries in analysis in the form of an anagram that deciphers to the sentence, ‘It is worthwhile to solve differential equations’. Accordingly, one can express the main idea behind the orbit method by saying ‘It is worthwhile to study coadjoint orbits’. - A.A. Kirillov

# Contents

# 1 Motivation - Back to the origins of GQ

The orbit method first developed by Kirillov refers to a collection of related procedures by which unitary representations of a Lie group $G$ can be constructed from certain geometric objects associated to $G$ called coadjoint orbits. Very roughly speaking, the idea behind is to unite harmonic analysis with symplectic geometry, so the method is a part of the more general idea to unite mathematics and physics. Each coadjoint orbit is a symplectic manifold and behaves somewhat like the classical phase space of a physical system having $G$ as a symmetry group. So we can consider the orbit method as an application of geometric quantization by appliying the geometric quantization procedure to every single orbit, conceived as a classical system, which we want to quantize. In this sense the orbit method tries to complete the circle: we understood geometric objects with group actions in terms of representations so far, and now we are going to understand representations in terms of geometric objects with group actions. And moreover: geometric quantization can be seen as a physical counterpart of the orbit method coming from pure mathematics.

But unlike you expect based on the order of the talks in this seminar, the orbit method leads us directly to the beginnings of geometric quantization and should therefore be one of the first talks in a seminar that follows the historical course: The modern theory of geometric quantization, which we considered in this seminar, was developed mainly by Kostant and Souriau in the 1970's. One of the motivations was to understand and generalize Kirillov's orbit method in representation theory, which he developed for nilpotent groups such as the Heisenberg group $H_3$ at the early 1960's. In the special case of nilpotent, simply connected groups he showed, using induced representations, that the theory produces a perfect correspondence between the set of coadjoint orbits of the group and its unitary dual $\hat{G}$, consisting of the unitary equivalence classes of irreducible unitary representations of G. Later, Kostant, Puánszky, Souriau and others extended the method (with some modifications) to solvable groups. More recently, David Vogan and others has studied the case of reductive groups. It turned out that the method not only gives a description of the unitary dual, but also gives simple and visual solutions to all other principal questions in representation theory: topological structure of the unitary dual, the explicit description of the restriction and induction functors, the formulae for generalized and infitesimal characters, the computation of the Plancherel measure, etc. Unfortunately these topics go to far to cover them in this talk.

Remains the question: Why is the orbit method called a method rather than a theorem? The main purpose of the orbit method is not explicitly to find all unitary representations of a given Lie group, but rather to find almost all representations of interest. Only for certain classes of Lie groups the method works perfeclty. Let us see more in detail what the orbit method is about and it's relationship to geometric quantization by first introducing the notion of coadjoint representations and orbits.

# 2 Coadjoint geometry

## 2.1 Coadjoint representations and orbits

We recover some basic theory for representations of Lie groups. Most of it was already introduced in talk 2 about symmetries in classical mechanics. For this chapter we follow Ref. [**1**].

**Reminder 1** *A representation of a Lie group $G$ is a Lie group homomorphism*

$$\pi : G \to \mathrm{Aut}(V),$$

*where $V$ is a vector space with dimension at least 1.*

*A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism*

$$\pi_* : \mathfrak{g} \to \mathfrak{gl}(V),$$

*where $\mathfrak{gl}(V)$ is the space $\mathrm{End}(V)$ with the commutator bracket.*

The vector space $V$ is called the representation space and a representation is said to be complex (real) if $V$ is a complex (real) vector space. As an example we consider the following inner automorphism on $G$:

**Example 1** *Let $G$ be a Lie group and define for each element $g \in G$ a map*

$$C_g : G \to G, \; x \mapsto gxg^{-1}.$$

*The differential of this map is the map $\mathrm{Ad}_g := (dC_g)_e \in \mathrm{End}(\mathfrak{g})$ and the mapping*

$$\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}), \; g \mapsto \mathrm{Ad}_g$$

*is called the* **adjoint representation***.*

In other words, the adjoint representation is a representation of a Lie group $G$ with its Lie algebra $\mathfrak{g}$ as representation space. In the case of a matrix Lie group $G$ the adjoint representation is simply the matrix conjugation. From the adjoint representation we can derive a further representation with representation space $\mathfrak{g}^*$:

**Definition 1** *Suppose $\pi$ is a Lie group representation acting on the vector space $V$. Then the dual representation $\pi^*$ acting on $V^*$ is given by*

$$\pi^*(g) = \pi(g^{-1})^t.$$

*If $\pi_*$ is a Lie algebra representation acting on $V$, then the dual representation is given by*

$$(\pi_*)^*(X) = -\pi_*(X)^t.$$

In particular, if we set $\pi = \mathrm{Ad}$ and $V = \mathfrak{g}$, we obtain a representation of the Lie group in $\mathfrak{g}^*$ that is dual to the adjoint representation in $\mathfrak{g}$. This is called the **coadjoint representation**. Since this notation is very important (but also for brevity:)) we use the special notation $K(g)$ for it, instead of the full notation $\mathrm{Ad}^*(g) = \mathrm{Ad}(g^{-1})^t$. So, by definition,

$$\langle K(g)F,\ X\rangle = \left\langle F,\ \mathrm{Ad}(g^{-1})X\right\rangle$$

where $X \in \mathfrak{g}, F \in \mathfrak{g}^*$, and by $\langle F,\ X\rangle$ we denote the value of a linear functional $F$ on a vector $X$. In case $G$ is a matrix Lie group, then $\mathfrak{g} \cong \mathfrak{g}^*$ and the coadjoint representation is just matrix conjugation again.

**Example 2** *If $\mathfrak{g}$ is semisimple, thus can be written as a direct sum of simple ideals, then the Killing form $\kappa$ is non-degenerate and we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the map*

$$F \mapsto X_F$$

*where $X_F$ is defined by*

$$F(Y) = \kappa(X_F, Y)\ \forall\ Y \in \mathfrak{g}.$$

*In this case the description of the coadjoint representation simplifies to*

$$F \mapsto \mathrm{Ad}(g^{-1})X_F.$$

Every representation defines a group action. If we define for $g \in G$, $F \in \mathfrak{g}^*$

$$g * F := K(g)F,$$

we easily see that we obtain a (smooth) group action of $G$ on $\mathfrak{g}^*$ via the coadjoint representation. If we consider the orbits under this group action, this leads us directly to the notion of a coadjoint orbit.

**Definition 2** *Let $F \in \mathfrak{g}^*$. The* **coadjoint orbit** *$\mathcal{O}_F$ of $F$ is the orbit of $F$ as $G$ acts on $\mathfrak{g}^*$ via the coadjoint representation, i.e.*

$$\mathcal{O}_F := G * F = \{g * F|\ g \in G\} \subset \mathfrak{g}^*$$

*Or equivalent: $\mathcal{O}_F$ is the image of the map $\kappa_F : G \mapsto \mathfrak{g}^*$ defined by $\kappa_F(g) = K(g)F$.*

The notion of a coadjoint orbit is the main ingredient of the orbit method and also the most important new mathematical object that has been brought into consideration in connection with the orbit method. As an example, we determine the coadjoint representation and orbits of SU(2), also known in physics as the spin group.

### 2.2 Coadjoint orbits of SU(2)

SU(2) is a matrix Lie group and can therefore be realized as a subgroup of $\mathrm{GL}(2,\mathbb{C})$.

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a,b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}$$

The corresponding Lie algebra $\mathfrak{su}(2)$ consists of the skew-hermitian $(2 \times 2)$ matrices with vanishing trace and can be written in the following manner:

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix} \,\middle|\, x,y,z \in \mathbb{R} \right\} \cong \mathbb{R}^3$$

We can identify $\mathfrak{su}(2)$ with $\mathfrak{su}(2)^*$ and therefore define a mapping $Q$ as

$$Q : \mathfrak{su}(2) \to \mathbb{C},\ F \mapsto \mathrm{tr}(F^2).$$

By applying the coadjoint action we obtain:

$$Q(K(g)F) = Q(gFg^{-1}) = \mathrm{tr}(gF^2g^{-1}) = \mathrm{tr}(F^2) = Q(F)$$

So $Q$ is invariant under the coadjoint action. This means that if you apply $Q$ on two elements of the same coadjoint orbit, you'll get the same complex value. Moreover, computing $Q(F)$ gives us

$$\begin{aligned} Q(F) &= \mathrm{tr}(F^2) \\ &= \mathrm{tr} \begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix} \cdot \begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix} \\ &= \mathrm{tr} \begin{pmatrix} -x^2-y^2-z^2 & \cdots \\ \cdots & -x^2-y^2-z^2 \end{pmatrix} \\ &= -2(x^2+y^2+z^2) = \text{const.} \end{aligned}$$

For this reason we can write $x^2 + y^2 + z^2 = \text{const.}/-2 = R^2$, for some positive real number $R$. If we have 2 points lying in a different sphere with 0 as the origin of the sphere, then $Q$ would have a different value for those points and they would therefore be in different coadjoint orbits. We conclude that 2 points of the same orbit will always lie in the same sphere. In the case $R = 0$, the point (0, 0, 0) forms a 0-dimensional coadjoint orbit on itself. Without proof we state that all the spheres exactly form all the coadjoint orbits of SU(2).

If one looks at the orbits just determined, one thing stands out: They all have even dimension. This is not accidental, but has a deep geometric reason, which we consider now.

## 2.3 Symplectic structure on coadjoint orbits

Our goal is to prove that all coadjoint orbits are symplectic manifolds and moreover, that each coadjoint orbit possesses a canonical G-invariant symplectic structure. This means that on each orbit $\Omega \subset \mathfrak{g}^*$ there is a canonically defined closed non-degenerate G-invariant differential 2-form $\omega_\Omega$.

**Reminder 2** *The stabilizer group of an element $F \in \mathfrak{g}^*$ under the coadjoint action is*

$$G_F = \{g \in G|\ g * F = F\} \leqslant G$$

*and is a Lie subgroup of $G$ due to the closed subgroup theorem.*

From the orbit-stabilizer and homogeneous-space construction theorem (Ref. [**11**]) we obtain

$$\mathcal{O}_F \cong G/G_F,$$

and that the coadjoint orbits are homogeneous spaces. For the definition of $\omega_\Omega$ we need the following Lemma:

**Lemma 1** *Denote with $\mathfrak{g}_F$ the Lie algebra of the stabilizing group $G_F$. Then the tangent space of $\mathcal{O}_F$ at $F$ is*

$$T_F\mathcal{O}_F \cong \mathfrak{g}/\mathfrak{g}_F.$$

*We can therefore consider elements of $T_F\mathcal{O}_F$ as elements of the form $X + \mathfrak{g}_F$ with $X \in \mathfrak{g}$.*

*Proof:* Since $\mathcal{O}_F$ is a homogeneous space, we consider the canonical projection $\pi_F : G \to G/G_F$. It holds $\pi_F(g) = K(g)F$, so the canonical projection corresponds to $K$ and $G$ can be considered as a fiber bundle over the base $\mathcal{O}_F$ with projection $\pi_F$. The fiber above $F$ is exactly $G_F$. By differentiating $\pi_F$ at $e$ we get a surjective map $(\mathrm{d}\pi_F)_e : \mathfrak{g} \to T_F(G/G_F)$. Consider the commutative diagram:

$$\begin{tikzcd} \ker((d\pi_F)_e) \arrow[r, hook] & \mathfrak{g} \arrow[r, "(d\pi_F)_e"'] \arrow[d] & T_F(G/G_F) \\ & \mathfrak{g}/\ker((d\pi_F)_e) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

By the homomorphism theorem, $\varphi$ is an isomorphism of vector spaces and we have $\mathfrak{g}/\ker((d\pi_F)_e) \cong T_F(G/G_F)$. But the kernel of $\pi_F$ consists of elements $g \in G$ such that $g * F = F$, so $\ker(\pi_F) = G_F$ and it holds: $\ker((d\pi_F)_e) = T_e(\ker(\pi_F)) = T_eG_F = \mathfrak{g}_F \Rightarrow T_F\mathcal{O}_F \cong T_F(G/G_F) \cong \mathfrak{g}/\mathfrak{g}_F$. □

Now we are ready to introduce an antisymmetric bilinear form $B_F$ on $\mathfrak{g}$ via the formula:

$$B_F(X, Y) := \langle F,\ [X, Y]\rangle$$

**Lemma 2** *The so called* **Kirillov-Kostant form** $B_F$ *is a well-defined, bilinear and skew-symmetric form on* $T_F\mathcal{O}_F$.

*Proof:* Since the Lie bracket is bilinear and skew-symmetric, it is sufficient to show that $B_F$ is well-defined. From the just proven Lemma 1 we know, that any representative of the coset $X+\mathfrak{g}_F$ can be written as $X+Z$ with $Z \in \mathfrak{g}_F$. By using the relation $\langle F,\ [X+Z,Y]\rangle = \langle F,\ [X,Y]\rangle + \langle F,\ [Z,Y]\rangle$, we have to show for the well-definedness that $[Z,Y] = \mathrm{ad}_Z(Y) = 0\ \forall\ Z \in \mathfrak{g}_F,\ Y \in \mathfrak{g}$, where ad denotes the adjoint representation of a Lie algebra. Using $\exp(Z),\ \exp(Z)^{-1} \in G_F\ \forall\ Z \in \mathfrak{g}_F$ we obtain for F:

$$(\exp(\mathrm{ad}_Z))^t F = (\mathrm{Ad}(\exp(Z)))^t F = \mathrm{K}(\exp(Z)^{-1})F = F \Rightarrow \exp(\mathrm{ad}_Z) = \mathrm{id}$$

This can be only fulfilled if $\mathrm{ad}_Z = 0 \Rightarrow \mathrm{ad}_Z(Y) = [Z,Y] = 0$ for all $Y \in \mathfrak{g}$. □

The short version: $B_F$ is well-defined since the form is invariant under $\mathfrak{g}_F$. We finally arrive at the definition of the symplectic form $\omega_{\mathcal{O}_F}$:

**Definition 3** *Let* $\mathcal{O}_F$ *be a coadjoint orbit in* $\mathfrak{g}^*$. *We define the 2-form* $\omega_{\mathcal{O}_F}$ *on* $\mathcal{O}_F$ *by*

$$\omega_{\mathcal{O}_F}(H)(\xi_X(H), \xi_Y(H)) = \omega_{\mathcal{O}_F}(H)(\mathrm{K}_*(X)H, \mathrm{K}_*(Y)H) := B_F(X,Y)$$

*for all* $X, Y \in \mathfrak{g},\ H \in \mathcal{O}_F$.

$\mathrm{K}_*$ denotes the infinitesimal version of the coadjoint action, i.e. the corresponding coadjoint representation $\mathrm{K}_* = d\mathrm{K}$ of the the Lie algebra $\mathfrak{g}$ with representation space $\mathfrak{g}^*$:

$$X\,\langle H, Y\rangle = \langle \mathrm{K}_*(X)H, Y\rangle = \langle H, -\mathrm{ad}_X Y\rangle = \langle H,\ [Y,X]\rangle$$

and it holds $\xi_X(H) = \mathrm{K}_*(X)H$, where $\xi_X$ is the vector field on $\mathcal{O}_F$ associated to X acting on smooth functions $f$ on $\mathcal{O}_F$ as

$$(\xi_X f)(H) := \frac{d}{dt}(f(\mathrm{K}(\exp(tX))H)).$$

**Theorem 1** *The tuple* $(\mathcal{O}_F, \omega_{\mathcal{O}_F})$ *is a G-invariant symplectic manifold. In particular, the form* $\omega_{\mathcal{O}_F}$ *is closed, hence defines on* $\mathcal{O}_F$ *a G-invariant symplectic structure.*

*Proof:* One can check $g^*\omega_{\mathcal{O}_F} = \omega_{\mathcal{O}_F}\forall g$. Since from above we know that $\mathcal{O}_F$ is a homogeneous space, it is enough to show that the 2-form $\omega_{\mathcal{O}_F}$ on $T_F\mathcal{O}_F$ is a symplectic form. From Lemma 2 we obtain the well-definedness, bilinearity and the skew-symmetry of $\omega_{\mathcal{O}_F}$.

Remains to show the non-degeneracy and closeness of the 2-form. We show these properties under the assumption that $G$ is a semisimple matrix Lie group. General proofs can be found in Ref. [**2**] using geometric observations, the notion of a Poisson manifold or the symplectic reduction procedure and are much more elegant than our proof here.

For a semisimple Lie group $G$ it holds $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ and therefore the center $\mathfrak{z} = \{X \in \mathfrak{g} \mid [X,Y] = 0 \ \forall\ Y \in \mathfrak{g}\}$ of its Lie algebra $\mathfrak{g}$ is trivial. Thus $\omega_{\mathcal{O}_F}$ is non-degenerate. We just need to prove the closeness.

$$\begin{aligned}
d\omega_{\mathcal{O}_F}(\xi_X,\xi_Y,\xi_Z) &= \xi_X\omega_{\mathcal{O}_F}(\xi_Y,\xi_Z) - \xi_Y\omega_{\mathcal{O}_F}(\xi_X,\xi_Z) + \xi_Z\omega_{\mathcal{O}_F}(\xi_X,\xi_Y)\\
&\quad - \omega_{\mathcal{O}_F}([\xi_X,\xi_Y],\xi_Z) + \omega_{\mathcal{O}_F}([\xi_X,\xi_Z],\xi_Y) - \omega_{\mathcal{O}_F}([\xi_Y,\xi_Z],\xi_X)\\
&= XB_F(Y,Z) - YB_F(X,Z) + ZB_F(X,Y)\\
&\quad - B_F([X,Y],Z) + B_F([X,Z],Y) - B_F([Y,Z],X)\\
&= X\langle F,[Y,Z]\rangle - Y\langle F,[X,Z]\rangle + Z\langle F,[X,Y]\rangle\\
&\quad - \langle F,[[Y,Z],X]\rangle + \langle F,[[X,Z],Y]\rangle - \langle F,[[Y,Z],X]\rangle\\
&= \langle F,[[Y,Z],X]\rangle - \langle F,[[X,Z],Y]\rangle + \langle F,[[X,Y],Z]\rangle\\
&\quad - \langle F,[[Y,Z],X]\rangle + \langle F,[[X,Z],Y]\rangle - \langle F,[[Y,Z],X]\rangle\\
&= 0 \text{ for all } X,Y,Z \in \mathfrak{g}.
\end{aligned}$$

We use that the vectors $\xi_X(F) = \mathrm{K}_*(X)F,\ X \in \mathfrak{g}$, span the whole tangent space $T_F\mathcal{O}_F$, since $G$ acts transitively on $\mathcal{O}_F$. Thus, $d\omega_{\mathcal{O}_F} = 0$. □

**Example 3** *As an example we consider the symplectic structure on the coadjoint orbits of SU(2), which are nested spheres: From definition 3 and theorem 1 we know that a symplectic structure on* $\mathbb{S}^2_r$ *with* $r > 0$ *is given by the Kirillov-Kostant form* $B_F$. *But what does* $\langle F,[X,Y]\rangle$ *for* $X,Y \in \mathfrak{su}(2)$ *and* $F \in \mathfrak{su}(2)^*$ *mean? To answer this we need a few statements about* $\mathfrak{su}(2)$ *and* $\mathfrak{so}(3)$*: It holds* $\mathfrak{su}(2) \cong \mathfrak{su}(2)^* \cong \mathfrak{so}(3)$ *and via the isomorphism*

$$\begin{pmatrix} 0 & -x_2 & x_3\\ x_2 & 0 & -x_1\\ -x_3 & x_1 & 0 \end{pmatrix} \mapsto (x_1,x_2,x_3),$$

*we obtain* $(\mathfrak{so}(3),[\cdot,\cdot]) \cong (\mathbb{R}^3,\times)$. *This allows us now to identify* $F$ *with* $X_F \in \mathfrak{so}(3)$ *via*

$$F(Y) = -\mathrm{tr}(X_F Y)/2 \ \forall\ Y \in \mathfrak{so}(3)$$

*and* $X_F$ *can in turn be identified with* $x_F \in \mathbb{R}^3$. *Therefore,* $[X,Y]$ *can be considered as a cross product and our symplectic structure can be expressed as*

$$\omega_{\mathbb{S}^2_r}(x)(u,v) = \langle x, u\times v\rangle / r^2 \text{ for } x \in \mathbb{S}^2_r,\ u,v \in T_x\mathbb{S}^2_r.$$

**Remark 1** *The Lie algebra structure of* $\mathfrak{g}$ *defines a canonical Lie-Poisson structure on* $\mathfrak{g}^*$ *by* $\{f,g\}(H) := \langle H,[df_H,dg_H]\rangle$ *for* $f,g \in C^\infty(\mathfrak{g}^*),\ H \in \mathfrak{g}^*$. *The mapping* $df_H : \mathfrak{g}^* \to \mathbb{R}$ *can be identified with an element of the bidual* $\mathfrak{g}^{**} \cong \mathfrak{g}$. *Therefore* $\mathfrak{g}^*$ *is a Poisson manifold. One can show that the symplectic leaves of the Poisson manifold* $(\mathfrak{g}^*,\{.,.\})$ *are exactly the coadjoint orbits. This was mentioned in talk 2 about symmetries in classical mechanics and can also be found in Ref. [**2**]. So we can take Theorem 1 as a corollary of this statement.*

# 3 Unitary representations from coadjoint orbits

## 3.1 The orbit method from the viewpoint of geometric quantization

Now that we know how to determine coadjoint orbits and that they are symplectic manifolds, the first thing that comes to mind is, that we can realize these manifolds as phase spaces. Finding unitary representations of $G$ then amounts to performing geometric quantization on quantizable orbits. At this point, we would like to briefly repeat the essential steps of the geometric quantization procedure for our case, so that we can apply them to SU(2) afterwards. We restrict ourselves to connected, simply connected, compact Lie groups following Ref. [**8**].

**Step 1**: Determine which of the coadjoint orbits $\mathcal{O}_F$ are quantizable. Thus, if we set $\hbar = 1$ the symplectic form $\omega_{\mathcal{O}_F}$ must fulfill the Bohr-Sommerfeld quantization condition:

$$\left[\frac{\omega_{\mathcal{O}_F}}{2\pi}\right] \in H^2_{dR}(\mathcal{O}_{\mathcal{F}}) \text{ lies in the image of } j : H^2(\mathcal{O}_{\mathcal{F}};\mathbb{Z}) \to H^2(\mathcal{O}_{\mathcal{F}};\mathbb{R}) \cong H^2_{dR}(\mathcal{O}_{\mathcal{F}}).$$

That means the integral of $\omega_{\mathcal{O}_F}$ over every closed 2 dimensional submanifold $S \subset \mathcal{O}_F$ is:

$$\int_S \omega_{\mathcal{O}_F} \in 2\pi\mathbb{Z}$$

**Step 2**: Choose a (Kähler) polarization $\mathcal{P}$ of $\mathcal{O}_F$.

**Step 3**: Construct a prequantum line bundle $(\mathcal{L}, \nabla, \langle\cdot,\cdot\rangle)$ over $(\mathcal{O}_F, \omega_{\mathcal{O}_F})$ with $\mathrm{curv}(\nabla) = \omega_{\mathcal{O}_F}$ and determine the Hilbert space $\mathcal{H}_P = L^2(\mathcal{L}) \cap \Gamma_{\mathcal{P}}(\mathcal{L})$ of square-integrable, polarized sections of $\mathcal{L}$. $\mathcal{H}_P$ will become our representation space.

**Step 4**: Determine the space $C^\infty_{\mathcal{P}}(\mathcal{O}_F)$ of polarization preserving functions on $\mathcal{O}_F$ and assign to each $f \in C^\infty_{\mathcal{P}}(\mathcal{O}_F)$ the Kostant-Souriau prequantum operator

$$\hat{f} = i\nabla_{X_f} + f.$$

**Step 5**: For $n = \dim\mathfrak{g}$, find a collection

$$f_1, \cdots, f_n \in C^\infty_{\mathcal{P}}(\mathcal{O}_F)$$

such that the set $\{\hat{f}_j\}$ is a linearly independent set of operators satisfying

$$[\hat{f}_i, \hat{f}_j] = \sum_{k=1}^{n} c^k_{ij} \hat{f}_k,$$

where $c^k_{ij}$ are the structure constants of $\mathfrak{g}$. Then $\{\hat{f}_j\}$ gives a symmetric representation $\pi_*$ of $\mathfrak{g}$ on $\mathcal{H}_{\mathcal{P}}$.

**Step 6**: Finally, lift $\pi_*$ to a unitary representation $\pi$ of $G$ on $\mathcal{H}_{\mathcal{P}}$. The finite-dimensional representations of a connected, simply connected Lie group are in one-to-one correspondence with finite-dimensional representations of its Lie algebra (cf. Ref. [**5**], Section 8.1.).

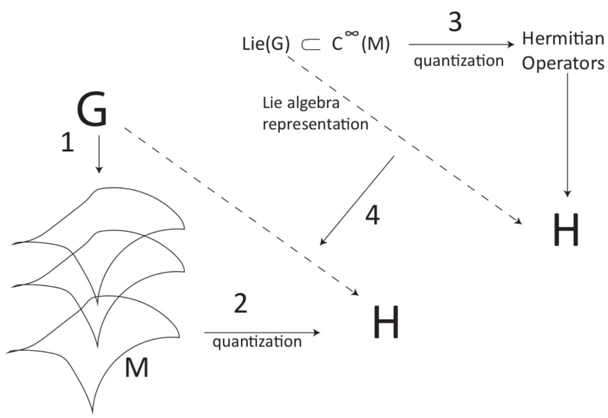


Figure 1: Philosophy: A Lie group $G$ acts on the coadjoint orbits (1), where one orbit $M$ is quantized (2). This gives rise to a Lie algebra representation (3), which in turn can be lifted to a unitary of the Lie group $G$ on $H$.

### 3.2 Unitary representations of SU(2)

Now we want to illustrate how the orbit method works in practice by applying the procedure just described for SU(2). To find the representations of SU(2), we need to quantize the coadjoint orbits, which we derived in section 2.2. The 0-dimensional coadjoint orbit which consists only of one single point gives us a trivial representation. Let us consider the 2-dimensional orbits $\mathbb{S}^2_r$ for $r > 0$.

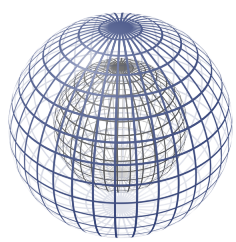

Figure 2: The coadjoint orbits of SU(2) are nested spheres

**Step 1**: Since the only closed 2-surface in $\mathbb{S}^2_r$ is the whole of $\mathbb{S}^2_r$ for $r > 0$ itself, the orbit $\mathbb{S}^2_r$ is quantizable if $\int_{\mathbb{S}^2_r} \omega_{\mathbb{S}^2_r} \in 2\pi\mathbb{Z}$. In section 2.3 we gave an expression for the symplectic form $\omega_{\mathbb{S}^2_r}$. Utilizing that $d\omega_{\mathbb{S}^2_r} = \frac{3}{r^2} dx \wedge dy \wedge dz$, we compute

$$\int_{\mathbb{S}^2_r} \omega_{\mathbb{S}^2_r} = \int_{\partial B^3_r} \omega_{\mathbb{S}^2_r} = \int_{B^3_r} d\omega_{\mathbb{S}^2_r} = \frac{3}{r^2} \int_{B^3_r} dx \wedge dy \wedge dz = 4\pi r.$$

Hence, $\mathbb{S}^2_r$ is quantizable if $r \in \mathbb{Z}/2$.

**Step 2**: Our aim is to find a Kähler structure on a general 2-sphere. If we consider the stereographic projection, we can express the symplectic form $\omega_{\mathbb{S}^2_r}$ as

$$\omega_{\mathbb{S}^2_r} = \frac{i}{r(1+|z|^2)^2} dz \wedge d\overline{z}.$$

If we define a Kähler potential $K$ in the following manner

$$K = \tfrac{1}{r}\log(1+|z|^2),$$

we obtain

$$i\partial\overline{\partial}K = \omega_{\mathbb{S}^2_r}.$$

So $\mathbb{S}^2_r$ is a Kähler manifold and hence admits the holomorphic and anti-holomorphic polarizations spanned by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$, respectively (cf. talk 8 about half-form corrections for Kähler polarizations).

**Step 3**: If we equip each coadjoint orbit $\mathbb{S}^2_r$ for $r \in \mathbb{Z}/2$ with the atlas given by the stereographic projection, one can construct a prequantum line bundle over $\mathbb{S}^2_r$. The Hilbert space of square-integrable, polarized sections of this line bundle is then the space $\mathcal{P}(\mathbb{C})$ of complex polynomials of degree $\leq$ 2r (cf. talk 8 about half-form corrections for Kähler polarizations).

**Step 4**: Expressing the potential 1-form $\theta$ and the Hamilton vector field $X_f$ for $f \in \mathcal{C}^\infty(\mathbb{S}^2)$ in stereographic coordinates, the Kostant-Souriau operator $\hat{f} = i\nabla_{X_f} + f$ becomes

$$\hat{f} = -r(1+|z|^2)^2\left(\frac{\partial f}{\partial \overline{z}}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \overline{z}}\right) - i\overline{z}(1+|z|^2)\frac{\partial f}{\partial \overline{z}} + f.$$

**Step 5**: It holds dim $\mathfrak{su}(2) = 3$. We want to find functions $f_1, f_2, f_3$ such that

$$[\hat{f}_i, \hat{f}_j] = c^k_{ij}\hat{f}_k,$$

where $c^3_{12} = c^1_{23} = c^2_{31} = 2$ are the structure constants of $\mathfrak{su}(2)$. One can check, that the following triple of functions does the job:

$$f_1 = \frac{i(z+\overline{z})}{r(1+|z|^2)},\ f_2 = \frac{z-\overline{z}}{r(1+|z|^2)},\ f_3 = \frac{-i(|z|^2-1)}{r(1+|z|^2)}$$

**Step 6**: We know that the Hilbert space is the space $\mathcal{P}(\mathbb{C})$ of complex polynomials with degree less than $2r$. The homogeneous polynomials $\mathcal{P}_n(\mathbb{C})$ of degree $n \leq 2r$ form an invariant subspace of $\mathcal{P}(\mathbb{C})$. Hence, $\hat{f}_1, \hat{f}_2$ and $\hat{f}_3$ form a representation of $\mathfrak{su}(2)$ in $\mathcal{P}_n(\mathbb{C})$. Lifting this to the group level gives us all irreducible, unitary representations of SU(2), which are well-know from the representation theory of compact groups.

### 3.3 The orbit method from the viewpoint of induced representations

We have seen so far, that in the representation theory of Lie groups one is mainly interested in $G$-invariant polarizations of homogenous symplectic $G$-manifolds, which are essentially coadjoint orbits. We are able to reduce the geometric and analytic problems to pure algebraic ones. We will now take a closer look at this and it will lead us directly to Kirillov's original consideration of the orbit method using the notion of induced representations. In the following we consider Lie groups, which are connected.

**Definition 4** *A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is* **subordinate** *to a functional $F \in \mathfrak{g}^*$ if the following equivalent conditions are satisfied:*

1. $F|_{[\mathfrak{h},\mathfrak{h}]} = 0$
2. *$X \mapsto \langle F, X\rangle$ is a 1-dimensional representation of $\mathfrak{h}$.*

Note that $\operatorname{codim}_{\mathfrak{g}}\mathfrak{h}$ is at least $\frac{1}{2}\operatorname{rank}(B_F)$ due to condition 1.

**Definition 5** *A subalgebra $\mathfrak{h}$ is a* **real algebraic polarization** *of $F$ if:*

1. *$\mathfrak{h}$ is subordinate to $F$.*
2. *$\operatorname{codim}_{\mathfrak{g}}\mathfrak{h} = \frac{1}{2}\operatorname{rank}(B_F)$, i.e. $\mathfrak{h}$ has maximal possible dimension.*

**Remark 2** *In the same way the notion of a* **complex algebraic polarization** *is defined by extending $F$ to the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ by complex linearity and consideration of complex subalgebras $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ that satisfy equivalent conditions as in the definition of a real algebraic polarization.*

**Definition 6** *An algebraic polarization $\mathfrak{h}$ is called* **admissible** *if it is invariant under the adjoint action of $G_F$, i.e. for any $g \in G_F$ we have:*

$$\operatorname{Ad}(g)X \in \mathfrak{h} \text{ for all } X \in \mathfrak{h}.$$

One can ask the question whether there exists a $G$-invariant polarization to a given F. This is not always the case, e.g. in the case $G = \mathrm{SU}(2)$, $\mathfrak{su}(2)$ has no subalgebras of dimension $2 = 3 - 2/2 = \dim\mathfrak{g} - \operatorname{rank}(B_F)/2$. A sufficient condition for this question can be found in Ref. [**2**].

We now establish a relationship between algebraic polarizations just introduced and geometric polarizations which we know from geometric quantization.

**Theorem 2** *There is a bijection between the set of all $G$-invariant real polarizations $P$ of a coadjoint orbit $\mathcal{O}_F$ and the set of admissible real algebraic polarizations $\mathfrak{h}$ of a given element $F \in \mathcal{O}_F$. To a polarization $P \subset T\mathcal{O}_F$ corresponds the algebraic polarization $\mathfrak{h} = (\pi_F)_*^{-1}(P(F)) \subset \mathfrak{g}$.*

*Proof:* The statement is based on a more general result about homogeneous manifolds:

**Lemma 3** *Let $M = G/K$ be a homogeneous manifold. Then:*

1. *There is a one-to-one correspondence between $G$-invariant subbundles $P \subset TM$ and $K$-invariant subspaces $\mathfrak{h} \subset \mathfrak{g}$ containing $\mathfrak{k}$.*
2. *The subbundle $P$ is integrable if and only if the corresponding subspace $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$.*

A proof of Lemma 3 can be found in Ref. [**2**]. So let us apply this statement to our problem: For a given subbundle $P \subset T\mathcal{O}_F$ we define $\mathfrak{h}$ as proposed in Lemma 3 with $\mathcal{O}_F$ in the role of $M$. When we considered the symplectic structure on coadjoint orbits we saw that $\pi_F^*(e)\omega_{\mathcal{O}_F}(F) = B_F$. Therefore $P(F)$ is maximal isotropic with respect to $\omega_{\mathcal{O}_F}$ if the same is true for $\mathfrak{h}$ with respect to $B_F$. The remaining statements of Theorem 2 follow then directly from Lemma 3. $\square$

**Theorem 3** *There is a bijection between the set of all $G$-invariant complex polarizations $P$ of a coadjoint orbit $\mathcal{O}_F$ and the set of admissible complex algebraic polarizations $\mathfrak{h}$ of a given element $F \in \mathcal{O}_F$. To a polarization $P \subset T\mathcal{O}_F^{\mathbb{C}}$ corresponds the complex algebraic polarization $\mathfrak{h} = (\pi_F)_*^{-1}(P(F)) \subset \mathfrak{g}^{\mathbb{C}}$.*

*Proof:* Analogous to theorem 2 reformulated for the complex case.

So now that we have established a correspondence between the geometric and algebraic views, let us look at how to obtain unitary and irreducible representations (unirreps in the following) of a Lie group via the algebraic view. To understand (or even classify) unirreps we introduce the notion of induced representations, since unirreps of a suitable closed subgroup $H \subset G$ are much easier to consider. Indeed: All unitary representations of a semisimple Lie group $G$ are suitable quotients (so called Langland's quotients) of induced representations of a parabolic subgroup $P \subset G$. Also, induced representations play a fundamental role in the Mackey machinery for the classification of the unitary dual of the Poincaré group.

We consider the following setting: Let $G$ be a locally compact group, $H \subset G$ a closed subgroup, $q : G \to G/H$ the canonical quotient map and denote with $\sigma : H \to GL(\mathcal{H}_\sigma)$ an unitary representation of $\mathcal{H}$ on the Hilbert space $(\mathcal{H}_\sigma, \langle \cdot, \cdot \rangle_\sigma)$. Let $\mathcal{C}(G, \mathcal{H}_\sigma)$ be the space of continuous functions from $G$ to $\mathcal{H}_\sigma$. How to get a representation of whole G based on the representation $\sigma$ on H, is descriped in the following **induction process**:

**Step 1**: Define the space:

$$\mathcal{F} := \{f \in \mathcal{C}(G, \mathcal{H}_\sigma) \mid q(\mathrm{supp}(f)) \text{ is compact and } f(\xi x) = \sigma(\xi) f(x),\ x \in G,\ \xi \in H\}$$

Then every element of $\mathcal{F}$ has the form

$$f_\alpha(x) = \int_H \sigma(y)\alpha(xy) d\mu(y),\ x \in G$$

for some continuous $\alpha \in C_c(G, \mathcal{H}_\sigma)$ with compact support (cf. Ref. [**4**], Prop. 6.1). Here $\mu$ denotes the Haar measure on $G$, which is well-defined for our purposes, since every Lie group is a locally compact group.

**Step 2**: Act with $G$ on $\mathcal{F}$ by right translations

$$f \mapsto r_x(f),\ x \in G,$$

i.e. we obtain a representation of $G$. For a more explicit formula see Ref. [**2**].

**Step 3**: Assume $G/H$ admits an invariant measure $\mu$ (see remark 3 and 4 for our case of interest), then $\langle f(x), g(x)\rangle_\sigma$ depends only on the coset $q(x)$ of $x$, and we obtain a function in $\mathcal{C}_c(G/H)$ which can be integrated over $G/H$ using $\mu$:

$$\langle f, g\rangle = \int_{G/H} \langle f(x), g(x)\rangle_\sigma d\mu(xH)$$

Since $\mu$ is invariant under $r_x$, this is an inner product invariant under $r_x$.

**Step 4**: Take the Hilbert space completion $\overline{\mathcal{F}}$ of $\mathcal{F}$. Then $r_x$ extends to an unitary representation of $G$ on $\overline{\mathcal{F}}$. We will denote the induced unitary representation by $\mathbf{Ind}_H^G(\sigma)$

**Remark 3** *Even if $G/H$ admits no invariant measure, we can modify the above process suitably to construct unitary representations. There is a $G$-invariant measure on $G/H$ if, and only if, $\Delta G|_H = \Delta H$ (where $\Delta G$ is the so-called modular function of $G$, which measures the difference between the left and right Haar measures). Further information can be found in Ref. [**4**], Ch. 8.*

By using the induction process, we are now going to describe **Kirillov's orbit method procedure** for the case of a connected, simply connected, nilpotent Lie group $G$. A similar procedure can be obtained for the case of compact instead of nilpotent groups using the notion of algebraic polarizations introduced above (cf. Ref. [**8**]).

**Definition 7** *A Lie group is nilpotent if its Lie algebra is nilpotent, where a Lie algebra $\mathfrak{g}$ is called nilpotent if there exists a decreasing finite sequence $(g_i)_{i\in[0,n]}$ of ideals such that*

$$\mathfrak{g}_0 = \mathfrak{g},\ \mathfrak{g}_n = 0 \text{ and } [\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}\ \forall\ i \in [0, n-1].$$

Somewhat more descriptively, nilpotent groups are just those groups that have a matrix realization by upper triangular matrices with ones on the diagonal. The nilpotent case is the simplest non-trivial case one can consider, e.g. for the nilpotent case a real $G$-invariant polarization always exists for a given $F$ in a coadjoint orbit $\Omega$ and all coadjoint orbits are integral. To formulate the orbit method procedure in the nilpotent case we still need a few statements about these kind of groups:

**Remark 4** *Let $G$ be a connected, simply connected, nilpotent Lie group. Then:*

1. $\exp$ *is a diffeomorphism that establishs a bijection between subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and closed connected subgroups $H \subset G$.*
2. *$G$ is unimodular with Haar measure given by the Lebesgue measure. In particular, any closed connected subgroup $H \subset G$ is unimodular, so $G/H$ admits an invariant measure (cf. remark 3).*

The second statement allows us to use the induction process from above. Further statements about nilpotent groups can be found in Ref. [**2**]. Moreover, on connected subgroups $H \subset G$ of nilpotent Lie groups we have the following statement about multiplicative characters:

**Remark 5** *Any multiplicative character of a connected Lie subgroup $H \subset G$ has the form*

$$\rho_{F,H}(expX) = e^{2\pi i \langle F,X \rangle},$$

*where $F \in \mathfrak{g}^*$ is a linear functional, such that $\mathfrak{h}$ is subordinate to $F$.*

Due to the Baker-Campbell-Hausdorff formula and that $\mathfrak{h}$ is subordinate to $F$, $\rho_{F,H}$ is a representation. A complete proof of this statement can be found in Ref. [**7**]. With this, the following steps now make sense:

**Step 1**: Pick a point $F \in \Omega$ in a coadjoint orbit $\Omega \subset \mathfrak{g}^*$.

**Step 2**: Find a real algebraic polarization $\mathfrak{h} \subset \mathfrak{g}$, i.e. a subalgebra of maximal dimension which is subordinate to $F$ ($F|_{[\mathfrak{h},\mathfrak{h}]} = 0$).

**Step 3**: Take $H := \exp\mathfrak{h}$ as a closed connected subgroup of $G$ (cf. Ref. [**2**], Ch. 2, Prop. 3) and consider the 1-dimensional unitary representations $\rho_{F,H}$, which has according to remark 5 the form

$$\rho_{F,H}(\exp X) = e^{2\pi i \langle F,X \rangle}.$$

**Step 4**: Induce these representations $\rho_{F,H}$ via the induction process described above to unitary representations $\mathrm{Ind}_H^G(\rho_{F,H})$ of $G$. For the case of connected, simply connected, nilpotent groups $G$, $\mathrm{Ind}_H^G(\rho_{F,H})$ is indeed an unirrep of $G$.

What can we say about the induced representations obtained in this way?

**Theorem 4** *Any obtained unitary representation depends neither on $F$ nor on the choice of the real algebraic polarization $\mathfrak{h}$. It depends only on the coadjoint orbit where $F$ lies in. Moreover, all unitary representations of $G$ are given by this coadjoint orbit procedure.*

So in the case of a connected, simply connected, nilpotent Lie group $G$ there is a bijection between the set of coadjoint orbits and the unitary dual of $G$. Kirillov was the first who proved this in his famous paper about the orbit method (Ref. [**1**]), and further proofs or modifications of this statement can be found in Ref. [**2**] or [**4**]. They all have in common that they use an induction procedure over the dimension of the group $G$ and make a case distinction between groups with 1-dimensional and groups with higher-dimensional center.

### 3.4 Unitary representations of the Heisenberg group

Let us now give a complete description of the unitary dual of the Heisenberg group $H_3$. The Heisenberg group is the simplest non-abelian nilpotent Lie group and actually the only one of dimension 3. Thus, from theorem 4 we know that via the orbit method we obtain a complete description of the unitary dual of $H_3$. From previous talks we know that $H_3$ and its Lie algebra $\mathfrak{h}_3$ can be written as:

$$H_3 = \left\{ g_{a,b,c} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a,b,c \in \mathbb{R} \right\}; \mathfrak{h}_3 = \left\{ xX+yY+zZ = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x,y,z \in \mathbb{R} \right\}$$

The adjoint action of $G$ on $\mathfrak{g}$ is given by matrix conjugation, so we get

$$\mathrm{Ad}(g_{a,b,c}) : (x,y,z) \mapsto (x,y,z-bx+ay).$$

If we identify an element $F = (\mu,\nu,\lambda) : (x,y,z) \mapsto \mu x + \nu y + \lambda z \in \mathfrak{h}_3^*$ with a strictly lower triangular matrix via

$$\begin{pmatrix} 0 & 0 & 0 \\ \mu & 0 & 0 \\ \lambda & \nu & 0 \end{pmatrix},$$

one can also calculate the coadjoint action

$$\mathrm{K}(g_{a,b,c}) : (\mu,\nu,\lambda) \mapsto (\mu + b\lambda, \nu - a\lambda, \lambda).$$

One can now easily deduce the coadjoint orbits of $H_3$. If $\lambda \neq 0$ then we can obviously choose appropriate $a$ and $b$ to map $(\mu,\nu,0)$ to $(0,0,\lambda)$. If $\lambda = 0$ then $(\mu,\nu,0)$ is $H_3$-stable. So we have two classes of orbits, a 1-parameter family $\Omega_\lambda$ where $\lambda \in \mathbb{R} \setminus \{0\}$ and a 2-parameter family $\Omega_{\mu,\nu}$ where $\mu,\nu \in \mathbb{R}$. The family $\Omega_\lambda$ consists of $z = \lambda$ planes, while the other family $\Omega_{\mu,\nu}$ consists of points in the $z = 0$ plane. Since $H_3$ is nilpotent, all orbits are integral and we can skip the calculation of the Kirillov-Kostant form. Let us calculate the unitary representations corresponding to the coadjoint orbits.

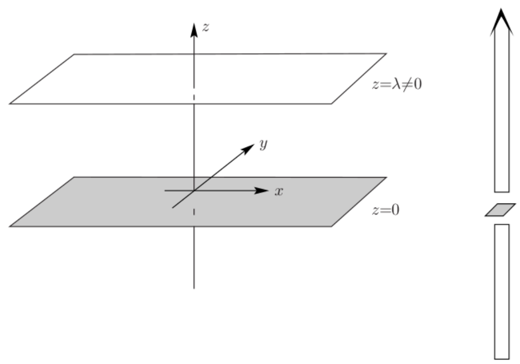


Figure 3: The coadjoint orbits of the Heisenberg group $H_3$

Consider $F = (\mu, \nu, 0) \in \Omega_{\mu,\nu}$. The kernel of $F$ is $\mathbb{R}Z \subset \mathfrak{h}_3$. So we need to find a maximal $\mathfrak{h} \subset \mathfrak{h}_3$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathbb{R}Z$. $\mathfrak{h} = \mathfrak{h}_3$ will do the trick. The corresponding unitary representation of $H = H_3$ is 1-dimensional:

$$\pi_{\mu,\nu}(\exp(X)) = e^{2\pi i \langle F, X\rangle} = e^{2\pi i(\mu x + \nu y)}$$
$$\Rightarrow \pi_{\mu,\nu}(g_{a,b,c}) = e^{2\pi i \langle F, X\rangle} = e^{2\pi i(\mu a + \nu b)}$$

Now consider the 1-dimensional orbits. Let $F = (0, 0, \lambda) \in \Omega_{\nu}$. The kernel of $F$ is $\mathbb{R}span\{X, Y\} \subset \mathfrak{h}_3$. This time the polarization $\mathfrak{h}$ cannot be all of $\mathfrak{h}_3$, so $\mathfrak{h} = \mathbb{R}span\{Y, Z\}$ is a possible choice. The corresponding unitary representations of $H = \{g_{0,b,c}\}$ is:

$$\rho(\exp(X)) = e^{2\pi i \langle F, X\rangle} = e^{2\pi i \lambda z}$$
$$\Rightarrow \rho(g_{0,b,c}) = e^{2\pi i \lambda c}$$

We need to induce this up to a representation of $H_3$. Following our induction process, we consider functions satisfying:

$$f(hg) = \rho(h)f(g) \text{ for } h \in H = \exp\mathfrak{h},\ g \in H_3.$$

This takes the form

$$f(g_{a,b,c}) = e^{2\pi i \lambda c} h(a),$$

for some smooth function $h : \mathbb{R} \to \mathbb{C}$ and gives us by right translation the representation:

$$(\pi_\lambda(g_{a,b,c})h)(t) = e^{2\pi i \lambda(c+bt)} h(t + a)$$

So we recover the Schrödinger representation (see last talk), and from the Stone-von Neumann theorem we know that any representation obtained via some other polarization is unitarily equivalent to this representation.

To sum up, for $H_3$ we established the correspondence:

$$\Omega_\lambda \leftrightarrow \pi_\lambda,\ \Omega_{\mu,\nu} \leftrightarrow \pi_{\mu,\nu}$$

between all coadjoint orbits and the unitary dual of $H_3$.

**Remark 6** *We have also seen in the last talk that the group multiplication of the Heisenberggroup in $n$ dimensions can be expressed in terms of vector addition on $\mathbb{R}^{2n+1}$. This means that the left-invariance of the Haar measure on $H_{2n+1}$ is equivalent to the left-invariance of the measure under translations. Therefore exists a constant $C > 0$ such that the left invariant Haar measure of the Heisenberggroup coincides with the $2n+1$-dimensional Lebesgue measure up to $C$. For the sake of convenience we set $C = 1$.*

# 4 Outlook: The orbit method for general types of groups

Unless you expect from the nilpotent case just considered, there is no general theory for the orbit method for arbitrary groups, but it can suggests the right answer and give in some cases a visual and adequate description of the situation. This includes the following important cases of Lie groups.

**Nilpotent groups**: As we stated in Theorem 4, the orbit method produces for connected, simply connected, nilpotent Lie groups all unirreps and thus establish a perfect correspondece between all coadjoint orbits and the unitary dual of the group.

**Exponential groups**: Such as in the nilpotent case the method produces here also all unirreps of the group, if it is connected and simply connected. Only the construction procedure has to be adapted, e.g. by the so called Pukanszky condition (cf. Ref. [**2**]).

**Solvable groups**: For general solvable, non exponential groups, with some modifications analogous to the exponential case, the method works also perfectly, if the group is connected and simply connected.

**Connected, compact groups**: In this case the method produces all unirreps of the group. There are strong conncections to the Borel-Weil-Bott theory. It is appropriate to regard geometric quantization as a generalization of the Borel-Weil-Bott theorem and the orbit method to the non-homogeneous case.

**Complex, semisimple groups**: The orbit method produces principal and degenerate series.

**SL(**$2, \mathbb{R}$): The orbit method produces discrete and principal series.

**Metaplectic group**: The orbit method produces the oscillator representations.

As you can see from the abundance of examples, the orbit method is a very useful tool for finding unitary representations of general Lie groups in the first place, even if there is no one to one correspondence as in the Heisenberg group case. And if the orbit method does not produce all unirreps, it gives us nevertheless new insights in the geometric structure. As usual, the faults of the method are the continuations of its advantages. We provide a brief overview of the merits and demerits of the orbit method following Ref. [**2**].

| | |
|---|---|
| 1. Universality: the method works for Lie groups of any type over any field. | 1. The recipes are not accurately and precisely developed. |
| 2. The rules are visual, easy to memorize and illustrated by a picture. | 2. Sometimes they are wrong and need corrections or modifications. |
| 3. The method explains some facts which otherwise look mysterious. | 3. Have fun translating this into a proof! |
| 4. It provides a great amount of symplectic manifolds and Poisson commuting families of functions. | 4. Most of the completely integrable dynamical systems were discovered earlier by other methods. |
| 5. The method introduce new fundamental notions: coadjoint orbit and moment map (we have not treated this in our talk.) | 5. The description of coadjoint orbits and their structures is sometimes not an easy problem. |

Finally, let us take a more abstract look at why the orbit method works from a physical viewpoint: On the classical level the phase space of a physical system with a given symmetry group $G$ is a symplectic $G$-manifold $M$. For an elementary system, i.e. a system that cannot be decomposed into smaller parts without breaking the symmetry, this manifold $M$ must be homogeneous. On the quantum level the phase space of a physical system with given symmetry group $G$ is a projectivization of a Hilbert space $\mathcal{H}$ with a unitary representation of $G$ on $\mathcal{H}$. For an elementary system this representation must be irreducible. Thus, the quantization principle suggests a correspondence between homogeneous symplectic $G$-manifolds on the one hand and unirreps of $G$ in the other. But actually the energy function for classical systems is defined up to an additive constant, while for a quantum system the energy is uniquely defined and is usually non-negative. This shows that the right classical counterpart to quantum systems with the symmetry group $G$ are Poisson $G$-manifolds rather than symplectic ones. But one can show that homogeneous Poisson $G$-manifolds are essentially coadjoint orbits. So we come to the desired correspondence between orbits and representations.

A further remark on this correspondence: One of the important unsolved problems in mathematics is the description of the unitary dual, the effective classification of irreducible unitary representations of all real reductive Lie groups. You see, these are topics of current research!

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